Reference Dependent Preferences and the EPK Puzzle

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Risk neutral valuation

- Arbitrage free market: there exists at least one state price density (SPD) i.e. a positive random variable π s.t.

$$\mathsf{E}[\pi] = 1.$$

- **Risk neutral valuation** at time 0 of random payoffs $\psi(S_T)$

$$\mathsf{E}\left[e^{-Tr}\pi\psi(S_T)\right] = \mathsf{E}\left[e^{-Tr}\,\mathsf{E}\left[\pi|S_T\right]\psi(S_T)\right],$$

$Tr = \int_0^T r_t dt$, $\{r_t\}_{t\in[0,T]}$ risk free rate, $\{S_t\}_{t\in[0,T]}$ risky asset.

Pricing Kernel

- Pricing kernel (PK) w.r.t. π, positive random variable

$$\mathcal{K}_\pi(S_T) = \mathsf{E}[\pi|S_T]$$

- Radon-Nikodym derivative of the risk neutral distribution Q w.r.t. physical measure P of S_T

$$Q(S_T \le x) \stackrel{\text{def}}{=} \int_0^x \mathcal{K}_\pi(s_T) dp(s_T) ds_T.$$

Intertemporal Pricing Kernel

- Conditional risk neutral measure $Q_t(S_T) = Q(S_T|\mathcal{F}_t)$

$$Q_t(S_T \leq x) \stackrel{\text{def}}{=} \int_0^x \mathcal{K}_\pi^t(ds_T) dp(s_T) ds_T.$$

for $P_t = Q(S_T|\mathcal{F}_t)$ and $\mathcal{F}_t = \{S_1, \dots, S_t\}$

- Intertemporal pricing kernel at time t (w.r.t. π)

$$\mathcal{K}_\pi^t(S_T) = \frac{q_t(S_T)}{p_t(S_T)}$$

q_t and p_t are cdtl pdf of Q_t and P_t respectively

Pricing Equation

Price at time t of a random payoff $\psi(S_T)$, $\tau = T - t$

- **Arbitrage free asset pricing models**

$$P_t = \mathrm{E}_t^P \left[e^{-\tau r} \psi(S_T) \frac{q_t(S_T)}{p_t(S_T)} \right]$$

- **Consumption based asset pricing models**

$$P_t = \mathrm{E}_t^P \left[\beta \frac{u'(S_T)}{u'(S_t)} \psi(S_T) \right]$$

β subjective discount factor, S_t value at time t of consumption, $u'(x)$ marginal utility index of the RA

Dual Nature of the PK

If (2) and (5) hold for any function ψ, the pricing kernel is

$$\mathcal{K}_\pi^t(S_T) = \frac{q_t(S_T)}{p_t(S_T)} = \frac{u'(S_T)}{u'(S_t)}$$

if $e^{-\tau r} = \beta$.

- Standard microeconomic theory
  - $u : \mathbb{R}_+ \rightarrow \mathbb{R}$
  - increasing, concave, twice cts. differentiable

$\rightarrow$ decreasing pricing kernel PK - Black-Scholes

Empirical Pricing Kernel (EPK)

- PK estimated from index options and prices: Ait-Sahalia & Lo (2000), Engle & Rosenberg (2002), Chernov (2003), Brown & Jackwerth (2004), Barone-Adesi, Engle & Mancini (2008), Giacomini & Härdle (2008), Bakshi, Madan & Panayotov (2010), Detlefsen, Härdle & Moro (2010), Chabi-Yo (2011), Christoffersen, Heston & Jacobs (2011), Audrino & Meier (2012), Grith, Härdle & Park (2013)

$\downarrow$

the puzzle

EPK puzzle

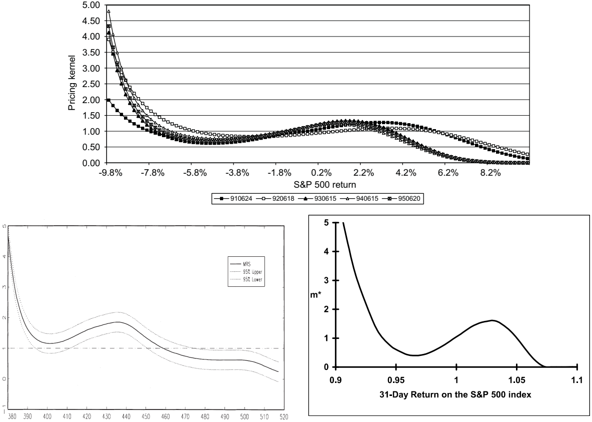


Figure 1: *S&P* 500 EPK's: Engle and Rosenberg (2002), Ait-Sahalia and Lo (2000), Brown and Jackwerth (2004)

EPK puzzle

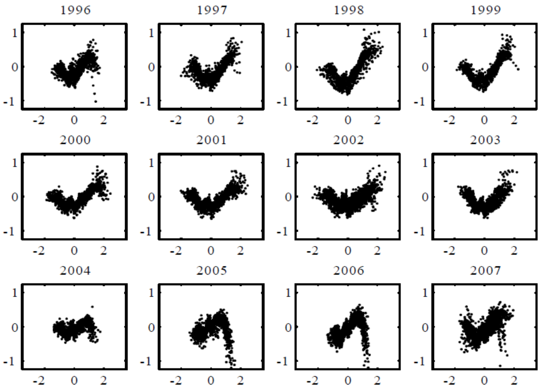


Figure 2: *S&P* 500 EPK's: Christoffersen, Heston and Jacobs (2012)

EPK puzzle

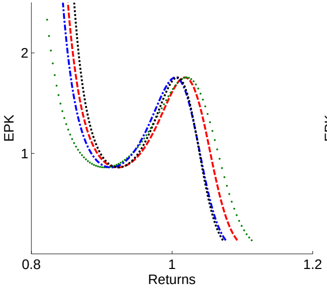


Figure 3: DAX EPK's for various maturities (left) and different estimation dates for fixed maturity 1M (right), Grith et al. (2013)

EPK puzzle

Figure 4: DAX 30 EPK's, Giacomini and Härdle (2008)

Empirical Tests for PK Monotonicity

Indirect estimation of the PK

$$\widehat{\mathcal{K}}_{\pi}^{t}(S_T) = \frac{\widehat{q}_t(S_T)}{\widehat{p}_t(S_T)}$$

- Golubev, Härdle and Timofeev (2008): LR test
- Härdle, Okhrin and Wang (2012): confidence bands for $\widehat{\mathcal{K}}_{\pi}^{t}$
- Beare and Schmidt (2012): concavity of the ordinal dominance curve associated with Q_t and P_t

Research Questions

- Consumption based asset pricing models

$$\mathcal{K}_{\pi}^{t}(\cdot) \propto u'(\cdot|\mathcal{F}_t)$$

representative agent with utility index u

- Can increasing regions in the PK be the outcome of investors' optimal behavior?
- How to modify standard preferences to rationalize the puzzle?
- What is the intuition behind the time variation of the PK w.r.t the new model?

Outline

1. Motivation ✓
2. Microeconomic Framework
3. Pricing Kernel
4. Comparative Statics
5. Fitting EPK's
6. Conclusions

Theoretical Explanation for the EPK puzzle

- **state dependence**: Benzoni, Collin-Dufresne & Goldstein (2005), Chabi-Yo, Garcia & Renault (2008), Christoffersen, Heston & Jacobs (2011)
- **heterogeneity in beliefs**: Ziegler (2007), Bakshi & Madan (2008), Bakshi, Madan & Panayotov (2010), Hens & Reichlin (2012)
- **misestimations/distortions**: Polkovnichenko & Zhao (2012), Hens & Reichlin (2012)
- **investors' sentiment**: Barone-Adesi, Mancini & Shefrin (2012)
- **ambiguity aversion**: Gollier (2011)
- **incomplete markets**: Hens & Reichlin (2012)
- . . .

Assumptions

- Financial markets
  - Finite investment time horizon $[0, T]$
  - Risk free bond $\{B_t\}_{0 \le t \le T}$ with annual interest rate r
  - Risky asset with prices $\{S_t\}_{0 \le t \le T}$ and return $R_T = S_T / S_0$
  - Arbitrage free market
  - No transaction costs; no restrictions on short sales

- m Consumers
  - Exogenous endowments w_0^i, $i = 1, \dots, m$
  - Decisions on portfolio holdings at $t = 0$
  - Financial wealth $e_i(R_T)$ and consumption $c_i(R_T)$

Individual Preferences

- Consumer i's **extended expected utility**, Mas-Colell et al. (1995)

$$\mathsf{E}\left[u^i\left\{R_T, c_i\left(R_T\right)\right\}\right],$$

with $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ - state dependent utility index

$$u^i\left\{R_T, c_i\left(R_T\right)\right\} = u_i^0\left\{c_i\left(R_T\right)\right\}\mathbf{I}\left\{R_T \in [0, x_i]\right\} + u_i^1\left\{c_i\left(R_T\right)\right\}\mathbf{I}\left\{R_T \in (x_i, \infty)\right\}$$

$x_i \in [0, \infty)$ - **reference point** of consumer i; $x_1 \leq \cdots \leq x_m$
u_i^0, $u_i^1 : \mathbb{R}_+ \to \mathbb{R}$ - utility indices
- strictly increasing, concave and twice cts differentiable

Equilibrium

- Individual optimization

$$\bar{c}_i\left(R_T\right) = \arg \max_{c_i(R_T)} \mathsf{E}\left[u^i\left\{R_T, c_i\left(R_T\right)\right\}\right]$$

$$\text{s.t. } \mathsf{E}\left[\left\{c_i(R_T) - e_i(R_T)\right\}\mathcal{K}(R_T)\right] \leq w_0^i$$

- Market clearing

$$\sum_{i=1}^{m} \bar{c}_i\left(R_T\right) = \sum_{i=1}^{m}\left\{w_0^i + e_i\left(R_T\right)\right\} \stackrel{\text{def}}{=} \bar{e}\left(R_T\right)$$

  - Pareto optimal $\bar{c}_1\left(R_T\right), \ldots, \bar{c}_m\left(R_T\right)$

Aggregated Preferences

- Aggregated extended expected preferences

$$\mathrm{E}\left[u_\alpha\left\{R_T, \bar{e}\left(R_T\right)\right\}\right],$$

with $u_\alpha : \mathbb{R}_+^2 \to \mathbb{R}$ - aggregated indirect utility

$$\begin{aligned} u_\alpha\left\{r_T, \bar{e}\left(r_T\right)\right\} = & u_{\alpha,1}\left\{\bar{e}\left(r_T\right)\right\} \mathbf{I}\left\{r_T \in [0, x_1]\right\} + \\ & + \sum_{i=1}^{m-1} u_{\alpha,i+1}\left\{\bar{e}\left(r_T\right)\right\} \mathbf{I}\left\{r_T \in (x_i, x_{i+1}]\right\} + \\ & + u_{\alpha,m+1}\left\{\bar{e}\left(r_T\right)\right\} \mathbf{I}\left\{r_T \in \left(x_m, \infty\right)\right\} \end{aligned}$$

for every realization r_T of R_T.

- Aggregated utility indices $u_{\alpha,j} : \mathbb{R}_+ \to \mathbb{R}$

$$
\begin{aligned}
u_{\alpha,j}\left\{\bar{e}\left(r_T\right)\right\} &= \sum_{k=1}^{m} \alpha_k u_k^0 \left\{\bar{c}_k\left(r_T\right)\right\} \mathbf{I}\left\{k \geq j\right\} \\
&+ \sum_{k=1}^{m} \alpha_k u_k^1 \left\{\bar{c}_k\left(r_T\right)\right\} \mathbf{I}\left\{k < j\right\}
\end{aligned}
$$

for for $j = 1, \dots, m+1$, importance weights $\alpha = (\alpha_1, \dots, \alpha_m)^\top$ and

$$
\frac{du_\alpha(r_T, \cdot)}{dy}\Big|_{y=\bar{e}(r_T)} = \alpha_i \frac{du^i(r_T, \cdot)}{dy}\Big|_{y=\bar{c}_i(r_T)}
$$

Pricing Kernel

Theorem

For every $\alpha_i > 0$ there exists β_i s.t.

$$
\begin{aligned}
\widetilde{\mathcal{K}}_\pi(r_T) = \alpha_i \beta_i \mathcal{K}(r_T) = \left.\frac{\partial u_{\alpha,1}\{y\}}{\partial y}\right|_{y=r_T} & \mathbf{I}\{r_T \in [0, x_1]\} + \\
+ \sum_{i=1}^{m-1} \left.\frac{\partial u_{\alpha,i+1}\{y\}}{\partial y}\right|_{y=r_T} & \mathbf{I}\{r_T \in (x_i, x_{i+1}]\} + \\
+ \left.\frac{\partial u_{\alpha,m+1}\{y\}}{\partial y}\right|_{y=r_T} & \mathbf{I}\{r_T \in (x_m, \infty)\}.
\end{aligned}
$$

for $\bar{e}(r_T) = r_T$.

Note: $\widetilde{\mathcal{K}}_\pi(r_T)$ is nonincreasing separately on the intervals $[0, x_1], (x_1, x_2], \ldots, (x_m, \infty)$ but may be nonmonotone at x_i's

Example 1

Consider m investors with identical reference point x_1 that switch between the v. Neumann-Morgenstern utility indices $u^0(y)$ and $u^1(y)$ s.t.

a. $\widetilde{\mathcal{K}}_\pi(r_T) = r_T^{-\gamma_\alpha^0}\,\mathbf{I}\{r_T \in [0, x_1]\} + r_T^{-\gamma_\alpha^1}\,\mathbf{I}\{r_T \in (x_1, \infty)\}$

b. $\widetilde{\mathcal{K}}_\pi(r_T) = r_T^{-\gamma_\alpha}\,\mathbf{I}\{r_T \in [0, x_1]\} + b r_T^{-\gamma_\alpha}\,\mathbf{I}\{r_T \in (x_1, \infty)\}$

γ_α^0, γ_α^1 and γ_α - aggr. CRRA coeff's, $b > 0$

Example 1

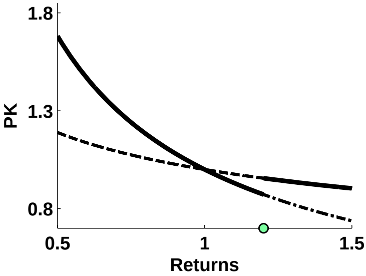


Figure 5: $\frac{du_\alpha^1(r_T)}{dr_T}$ (dashed-dotted) and $\frac{du_\alpha^{m+1}(r_T)}{dr_T}$ (dashed) for $x_1 = 1.2$; a. (left) $\gamma_\alpha^0 = 0.75 > \gamma_\alpha^1 = 0.25$; b. (right) $\gamma_\alpha = 0.50$ and $b = 1.2$

Example 2

Consider m investors with ref. points x_i's that switch between utility indices $u^0(y)$ and $u^1(y)$ with $u^1(y) = bu^0(y)$

$$u^0(y) = \begin{cases} \dfrac{y^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(y) & \text{if } \gamma = 1 \end{cases}$$

Let $F(r_T)$ be the cdf of the reference points

$$F(r_T) = m^{-1} \sum_{i=1}^{m} \mathbf{I}\{x_i \leq r_T\}$$

$$\widetilde{\mathcal{K}}_\pi(r_T) = \left[\frac{r_T}{1 + F(r_{T,t})\left(b^{\frac{1}{\gamma}} - 1\right)} \right]^{-\gamma} \tag{1}$$

Example 2

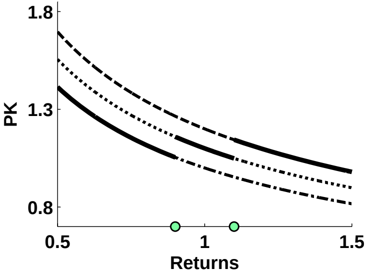


Figure 6: $\frac{du_\alpha(r_T)}{dr_T}$ (solid), $\frac{du_\alpha^j(r_T)}{dr_T}$ (dotted), $\frac{du_\alpha^1(r_T)}{dr_T}$ (dashed dotted) and $\frac{du_\alpha^{m+1}(r_T)}{dr_T}$ (dashed) for $\gamma_\alpha = 0.75$ and $b = 1.2$; $m = 3$ and $m = 5$

Example 2

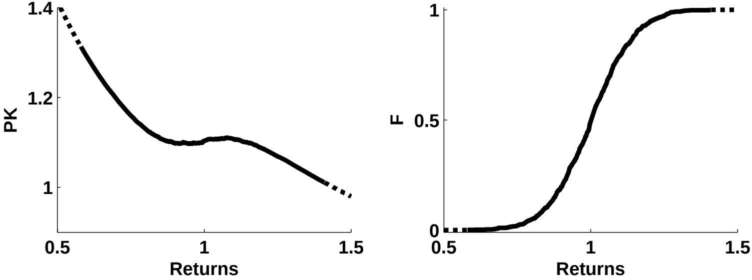


Figure 7: *PK* (left) for $\gamma = 0.5$, $b = 1.2$ and F (right) a *edf* of 400 random reference points from N$(1, 1.2)$; compact support for pdf of F (solid)

Welfare Effects

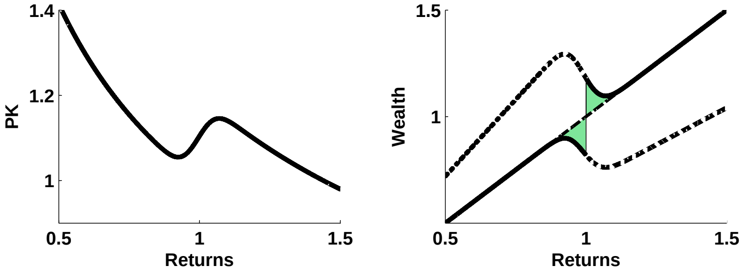


Figure 8: Market pricing kernel and (scaled) final wealth of a mixed agent; $m\bar{c}_i(r_T)$ (solid), $m\bar{c}_i^0(r_T)$ (dotted) and $m\bar{c}_i^1(r_T)$ (dotted)

Welfare Effects

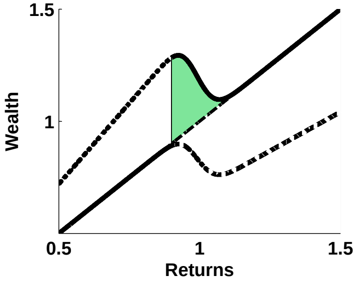


Figure 9: Final wealth of an optimistic agent (left) and pessimistic agent (right); $m\bar{c}_i(r_T)$ (solid), $m\bar{c}_i^0(r_T)$ (dotted) and $m\bar{c}_i^1(r_T)$ (dotted)

Effects of F

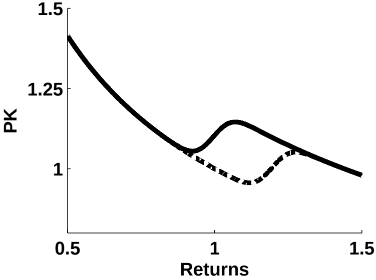


Figure 10: Baseline model (solid): $\gamma = 0.5$, $b = 1.2$, $F = \mathrm{N}(1, 0.05)$; alternative specifications (dashed): left $F = \mathrm{N}(1.2, 0.05)$; right $F = \mathrm{N}(1, 0.15)$

Effects of θ

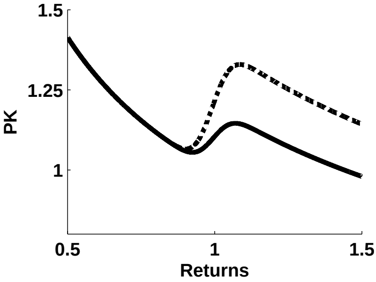


Figure 11: Baseline model (solid): $\gamma = 0.5$, $b = 1.2$, $F = \mathrm{N}(1, 0.05)$; alternative specifications (dashed): left $b = 1.4$; right $\gamma = 0.25$;

Fitting EPK's

- Assume that for the estimate $\widehat{\mathcal{K}}(s_j) = y_j$ with $j = 0, \ldots, n$

$$y_j = \mathcal{K}_{\theta,F}\left(s_j\right) + \varepsilon_j, \quad \text{with } \varepsilon_j \sim (0, \sigma^2)$$

$$\mathcal{K}_{\theta,F}\left(x\right) = \left[\frac{x}{\left\{1 - F\left(x\right)\right\} b_0^{\frac{1}{\gamma}} + F\left(x\right) b_1^{\frac{1}{\gamma}}}\right]^{-\gamma}$$

  with $b_1 = b_0 b > 0$, $\theta = \left(\gamma, b_0, b_1\right)^{\top}$ and F cdf.

- Find $\widehat{\theta}$ and $\widehat{F}$ that minimize

$$\sum_{j=1}^{n} \left\{y_j - \mathcal{K}_{\theta,F}\left(s_j\right)\right\}^2$$

Identifiability

For $\gamma, b_0, b_1 > 0$ and $b_0 \leq b_1$

$$x\mathcal{K}^{\frac{1}{\gamma}}_{\theta,F}(x) = \{1 - F(x)\}\, b_0^{\frac{1}{\gamma}} + F(x)\, b_1^{\frac{1}{\gamma}} \tag{2}$$

is a monotonically increasing function bounded between $b_0^{\frac{1}{\gamma}}$ and $b_1^{\frac{1}{\gamma}}$.

- For discrete reference points θ is identifiable
- For F continuous θ is not identifiable

Partial Identifiability

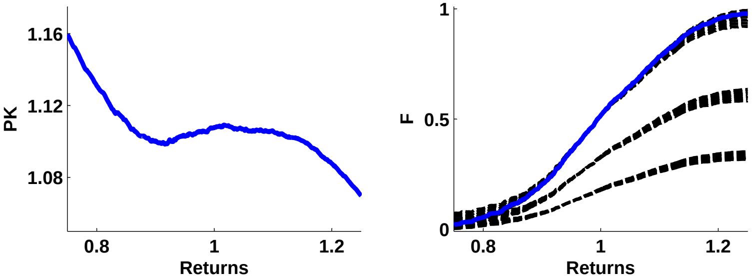
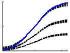


Figure 12: PK (left) and F (right) for $b = (1.2, 1.3, 1.5)$ and $\gamma = (0.46, 0.47, 0.48, 0.49, 0.50, 0.52)$



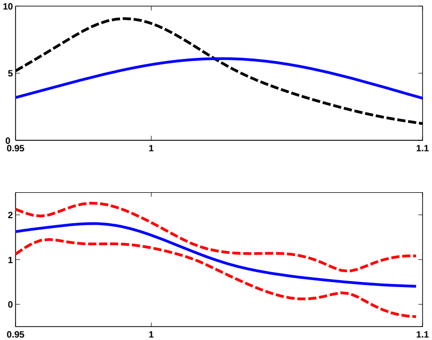


Figure 13: Upper panel: estimated risk neutral density $\widehat{q}$ and historical density $\widehat{p}$. Lower panel: EPK and 95% uniform confidence bands on 20080228, Härdle, Okhrin and Wang (2012)

EPK Dynamics

- Assume that $y_{tj} = \widehat{\mathcal{K}}^t(s_j)$ is a sample of T noisy curves

$$y_{tj} = \mathcal{K}_{\theta_t, F_t}(s_j) + \varepsilon_{tj} \quad \text{with } \varepsilon_{tj} \sim (0, \sigma_t^2)$$

$$\mathcal{K}_{\theta_t, F_t}(x) = \left[\frac{x}{\{1 - F_t(x)\}\, b_{0t}^{\frac{1}{\gamma_t}} + F_t(x)\, b_{1t}^{\frac{1}{\gamma_t}}} \right]^{-\gamma_t}$$

  with $\theta_t = (\gamma_t, b_{0t}, b_{1t})^\top$ and F_t cdf.

- Scale/shift model for F ▸ SIM EPK

$$F_t(x) = F\left(\frac{x - a_t}{d_t}\right) \quad \text{for} \quad a_t \in \mathbb{R} \quad \text{and } d_t \in \mathbb{R}_+$$

- use state variables to pin down $(\gamma_t, b_{0t}, b_{1t}, a_t, d_t)$ for parametric F

Other Setups I

- Option implied stock return distributions

$$p_t(S_{t+1}|\theta, F) = \frac{\frac{q_t(S_{t+1})}{\mathcal{K}_{\theta_t, F_t}(S_{t+1})}}{\int \frac{q(x)}{\mathcal{K}_{\theta_t, F_t}(x)}}$$

- Maximum likelihood estimation

$$(\widehat{\theta}, \widehat{F}) = \arg\min_{\theta, F} \sum_{t=0}^{T-1} \log p_t(S_{t+1}|\theta, F)$$

Other Setups II

- Euler equation

$$A_t = \mathrm{E}_t \left[e^{-r_{t,t+1}} \mathcal{K}_{\theta_t, F_t}(S_{t+1}) A_{t+1} \right], \quad \text{t=1,...,T}$$

$A_t = (A_{1t}, \ldots, A_{kt})^\top$ price vector of k assets at t

- Generalized method of moments

$$g_T(\theta, F) = \sum_{t=0}^{T-1} \left\{ e^{-r_{t,t+1}} \mathcal{K}_{\theta_t, F_t}(S_{t+1}) A_{t+1} / A_t - 1_k \right\}$$

$$(\widehat{\theta}, \widehat{F}) = \arg \min_{\theta, F} \left\{ g_T^\top(\theta, F) W^{-1} g_T(\theta, F) \right\}.$$

for some weighting matrix W.

Conclusions

- Individual state-dependent preferences with reference point may explain nonmonotonicity in the PK
- The model suggests a sensible mechanisms for PKs dynamics

Further Research

- Statistical estimation and inference on $\widehat{\theta}$ and $\widehat{F}$
- Joint fitting of curves using state variables
- Alternative model specifications for the EPK puzzle

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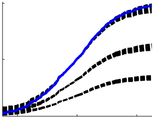

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PK under the Black-Scholes Model ▸ Motivation

- Geometric Brownian motion for S_t

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

μ drift, σ volatility, W_t Wiener process

- Physical density p is log-normal, $\tau = T - t$

$$p_t(S_T) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{1}{2}\left\{\frac{\log(S_T/S_t) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right\}^2\right]$$

- Risk neutral density q is log-normal with drift μ

PK under the Black-Scholes Model ▸ Motivation

- PK is a decreasing function in S_T for fixed S_t

$$\mathcal{K}_t\left(S_T\right)=\left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}}\exp\left\{\frac{(\mu-r)\left(\mu+r-\sigma^2\right)\tau}{2\sigma^2}\right\}$$
$$=c\left(\frac{S_T}{S_t}\right)^{-\gamma}=b\frac{u'(S_T)}{u'(S_t)}$$

- and consistent with a power utility function $u(x)=\frac{x^{1-\gamma}}{1-\gamma}$ for $c=\exp\left\{\frac{(\mu-r)\left(\mu+r-\sigma^2\right)\tau}{2\sigma^2}\right\}$ and $\gamma=\frac{\mu-r}{\sigma^2}\geq 0$ constant relative risk aversion (CRRA) coefficient.

EPK Dynamics - Empirical Study

- Grith, Härdle and Park (2013)
- **Data**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de
- Datastream DAX 30 Price Index;
  2 years worth of monthly returns in a sliding window
- EUREX European Option Data; tick observations;
  intraday cross-sectional data

Estimation of PK

- EPK: ratio of 2 estimated densities

$$\hat{\mathcal{K}}_t(S_T) = \frac{\hat{q}_t(S_T)}{\hat{p}_t(S_T)}$$

- $\hat{q}_t(S_T)$ by Rookley (1997) method based on the results of Breeden and Litzenberger (1978)

$$q_t(S_T) = e^{r\tau} \left. \frac{\partial^2 C_t(\cdot)}{\partial K^2} \right|_{K=S_T}$$

C European call price with strike price K

- $\hat{p}_t(S_T)$ by kernel density method

Estimation of RND

Rookley method: for fixed one month maturity estimate a smooth call price function with respect to the moneyness K/S_t

- implied volatility σ_{IV} substitute the call price
- $\hat{\sigma}_{IV}$, $\hat{\sigma}'_{IV}$, $\hat{\sigma}''_{IV}$ improve efficiency
- local polynomial smoothing of degree 3
- quartic kernel
- little sensitivity to the bandwidth choice

Estimation of PDF

- nonparametric kernel density based on overlapping monthly historical returns (2 years)
- quartic kernel
- bandwidth choice: unimodal densities for all periods
- peak varies with the bandwidth and window length
- robustness checks with risk-free mean adjusted historical densities and GARCH models with empirical innovations

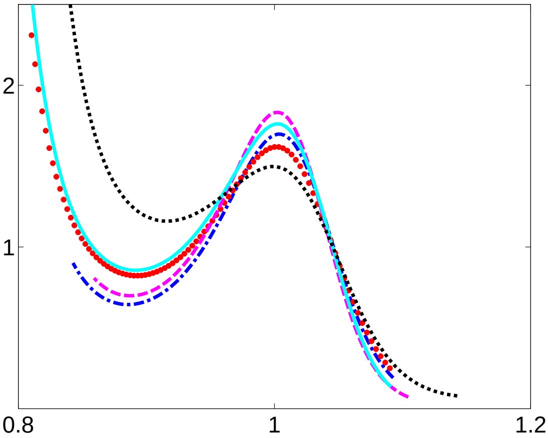

Figure 14: EPK vs moneyness for maturity τ =0.083 (4w), observed on 20060118 (blue), 20060215 (red), 20060322 (magenta), 20060419 (cyan), 20060417(black)

Shape Invariant Model (SIM)

- Y_{tj} is a noisy sample of T curves at design points u_j, with $j \in \{1, \ldots, n\}$, $n = 101$

$$Y_{tj} = \mathcal{K}_t(u_j) + \varepsilon_{tj} \,, \quad \text{with} \quad \varepsilon_{tj} \sim (0, \sigma_t^2) \tag{3}$$

- The smooth curves are of the form

$$\mathcal{K}_t(u) = \theta_{t1} \mathcal{K}_0 \left(\frac{u - \theta_{t3}}{\theta_{t2}} \right) + \theta_{t4} \tag{4}$$

- $\mathcal{K}_0$ is a reference curve and $\theta = (\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4})^\top$ are horizontal and vertical deviation parameters

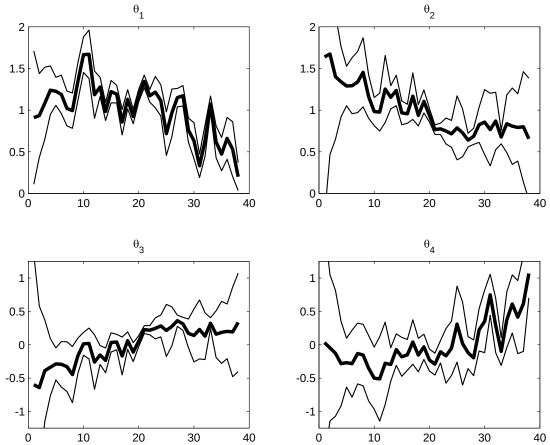

Figure 15: Parameter estimates of the SIM and their confidence intervals at 95% confidence level for the EPK 200304:200605

Business Cycle Indicators

- **Data**: Daily observations. German market
- Credit spread (CD): 5Y Corporate - Gov. bond yield
- Yield term slope (IR): 30Y-3M Gov. bond yield
- DAX 30 stock index (I_{DAX})

	θ_1	θ_2	θ_3	θ_4	CS	DAX	YT
θ_1	1.00	0.55*	0.02	0.78*	-0.25	0.38**	-0.26
θ_2		1.00	0.38*	-0.04	0.06	-0.12	-0.39**
θ_3			1.00	-0.18	0.07	-0.21	-0.28***
θ_4				1.00	-0.37**	0.62*	-0.04

Table 1: Correlation table for the first difference of SIM parametethers and the selected macro economic variables. (sig. at 1% = *, sig. at 5% = **, sig. at 10% = ***)

Interpretation

EPK Dynamics

When economic conditions deteriorate ...

SIM Model

... the hump moves to the right, its spread increases, its height decreases

Our Model

... the investors become more pessimistic: mean of F increases

... their heterogeneity increases: variance of F increases